Theorem 1 *Let* $A, B, C \in \mathbb{R}$. *Suppose* $\sum_{n=1}^{\infty} a_n = A$, $\sum_{n=1}^{\infty} b_n = B$, *and* $\sum_{n=1}^{\infty} c_n = C$, *where* $c_n = \sum_{k=1}^{n} a_k b_{n-k+1}$. *Then* $C = AB$.

Proof: We begin with the following claim.

Claim : Suppose u_n is a bounded sequence. Then $\sum_{n=1}^{\infty} u_n x^n$ converges absolutely for $x \in [0, 1)$.
Proof of Claim : There exists an M, such that for all $n \geq N$,

$$\sum_{k=1}^{n} | u_k x^k | \leq M \sum_{k=1}^{n} x^k \leq \frac{M}{1-x}.$$

The claim follows easily. □

By the above claim and the hypothesis of the Theorem, $h, g, f : [0, 1) \to \mathbb{R}$ given by

$$h(x) = \sum_{n=1}^{\infty} c_n x^{n+1}, \quad g(x) = \sum_{n=1}^{\infty} b_n x^n, \quad \text{and} \quad f(x) = \sum_{n=1}^{\infty} a_n x^n$$

are well-defined and by result proved in class,

$$h(x) = f(x)g(x), \quad \forall x \in [0, 1).$$

Let $\epsilon > 0$ be given. Then there exists an $N \in \mathbb{N}$ such that

$$\max\{|A_n - A|, |B_n - B|, |C_n - C|\} < \epsilon \quad \forall n \geq N,$$

where $A_n = \sum_{k=1}^{n} a_k$, $B_n = \sum_{k=1}^{n} b_k$, and $C_n = \sum_{k=1}^{n} c_k$. Observe that for any $x \in [0, 1)$,

$$\begin{aligned}
|f(x) - A| &= \left|\sum_{n=1}^{\infty} a_n x^n - A\right| = (1-x)\left|\sum_{n=1}^{\infty} (A_n - A)x^n - A\right| \quad \text{(why ?)} \\
&\leq (1-x)\left(\sum_{n=1}^{N} |(A_n - A)x^n| + \sum_{n=N+1}^{\infty} |(A_n - A)x^n| + |A|\right) \\
&\leq (1-x)\left(\sum_{n=1}^{N} |A_n - A| + |A|\right) + \epsilon. \quad \text{(why ?)}
\end{aligned}$$

Similarly for all $x \in [0, 1)$ we have that

$$\begin{aligned}
|g(x) - B| &\leq (1-x)\left(\sum_{n=1}^{N} |B_n - B| + |B|\right) + \epsilon, \\
|h(x) - xC| \leq |\sum_{n=1}^{\infty} c_n x^n - C| &\leq (1-x)\left(\sum_{n=1}^{N} |C_n - C| + |C|\right) + \epsilon.
\end{aligned}$$

So, for any $x \in [0,1)$ we have

$$\begin{aligned}
|C - AB| &= |C - Cx + Cx - h(x) + f(x)g(x) - AB| \\
&\leq |C - Cx| + |Cx - h(x| + |f(x| \; |g(x) - B| + |B| \; |f(x) - A| \\
&\leq (1 - x|C| + (1-x)\left(\sum_{n=1}^{N} |C_n - C| + |C|\right) + \epsilon + \\
&\quad \left[|A| + (1-x)\left(\sum_{n=1}^{N} |A_n - A| + |A|\right) + \epsilon\right]\left[(1-x)\left(\sum_{n=1}^{N} |B_n - B| + |B|\right) + \epsilon\right] + \\
&\quad + |B|\left[(1-x)\left(\sum_{n=1}^{N} |A_n - A| + |A|\right) + \epsilon\right] \\
&\leq (1-x)\left[|C| + \sum_{n=1}^{N} |C_n - C| + |C| + |A|\left(\sum_{n=1}^{N} |B_n - B| + |B|\right) + \right. \\
&\quad + \left(\sum_{n=1}^{N} |A_n - A| + |A|\right)\left(\sum_{n=1}^{N} |B_n - B| + |B| + \epsilon\right) + \\
&\quad \left. + \epsilon\left(\sum_{n=1}^{N} |B_n - B| + |B|\right) + |B|\left(\sum_{n=1}^{N} |A_n - A| + |A|\right)\right] \\
&\quad + \epsilon + |A|\epsilon + \epsilon^2 + |B|\epsilon \\
&= (1-x)C_1 + C_2\epsilon + \epsilon^2,
\end{aligned}$$

where $C_1 \equiv C_1(N, A_n, B_n, C_n, A, B, C)$, $C_2 \equiv C_2(A, B)$. As $x \in [0,1)$ was arbitrary,(and choice of N does not depend on x), we can take $x = 1 - \frac{\epsilon}{1+C_1+\epsilon}$. Therefore, we have

$$|C - AB| < C_4\epsilon + \epsilon^2.$$

As $\epsilon > 0$ was aribtrary we can conclude that $C = AB$ (why?). $\square$