Problems to be turned in: 1
Due: 25th, October 2008

1. Show that $(L^\infty(\Omega, \mathcal{B}, \mu), ||\cdot||_\infty)$ is a complete normed vector space.

2. Let $(\Omega, \mathcal{B}, \mu)$ be a finite measure space. If $1 \leq s \leq r \leq \infty$, then $L^r(\Omega, \mathcal{B}, \mu) \subset L^s(\Omega, \mathcal{B}, \mu)$; in fact, we have
$$||\phi||_s \leq (\mu(\Omega))^{\frac{1}{s}-\frac{1}{r}} \ ||\phi||_r \ \forall \ \phi \ .$$

3. Let $-\infty \leq a < b\infty$. A function $\phi : (a, b) \to \mathbb{R}$ is said to be convex if
$$\phi((1-\lambda)x + \lambda y) \leq (1-\lambda)\phi(x) + \lambda\phi(y)$$
for all $a < x, y < b$ and $0 \leq \lambda \leq 1$.
   (a) Show that ϕ is continuous.
   (b) Show that $\frac{\phi(t)-\phi(s)}{t-s} \leq \frac{\phi(u)-\phi(t)}{u-t}$ whenever $a < s < t < u < b$.
   (c) Let $(\Omega, \mathcal{B}, P)$ be a probability space. Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function. If $X : \Omega \to \mathbb{R}$ is integrable then show that
$$\phi(\int XdP) \leq \int (\phi(X))dP.$$
   (*Hint:* Set $t = \int Xdp$, $s = X(\omega)$ and use the above step)