Problems to be turned in: 4,6,7
Due: October 15th, 2008

1. Extend the notion of product measure to $\prod_{i=1}^{n}(\Omega_i, \mathcal{B}_i, \mu_i)$.

2. Let $\mathcal{C} = \{A : A \subset \mathbb{R}$ and A is countable or A^c is countable $\}$. Show that $\mathcal{C}$ is a σ algebra and that $D = \{(x, x) : x \in \mathbb{R}\}$ does not belong to $C \otimes C$ although all its sections D^x and D_y belong to $\mathcal{C}$.

3. We constructed the product measure for finite measure spaces in class. Deduce from this that the product measure for σ finite measure spaces also exist.

4. Show that if f is integrable on $(\Omega, \mathcal{B}, \mu)$, then
$$\lim_{\mu(B)\to 0} \int_B f d\mu = 0$$
(i.e given any $\epsilon > 0$, there is $\delta > 0$ such that if $B \in \mathcal{B}$ and $\mu(B) < \delta$, then $| \int_B f d\mu |< \epsilon$).

5. Following the notation in class, show that the sets E_+^-, E_-^-, E_+^+ have lebesgue measure zero.

6. Show that if $f \in L^1([a, b], \lambda)$, then $F(x) = \int_a^x f(y)dy$ is an absolutely continuous function.

7. Let C be the Cantor set.
   (a) Show that if $x \in C$ then $x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}$ where $a_j = 0$ or $a_j = 2$ for all j.
   (b) Define a function $f : C \to [0, 1]$ as follows:
   $$f(x) = \sum_{j=1}^{\infty} \frac{b_j}{2^j},$$
   where $x = \sum_{j=1}^{\infty} \frac{a_j}{3^j}$ and $b_j = \frac{a_j}{2}$.
      i. Show that f maps C onto $[0, 1]$.
      ii. If $x, y \in C$, $x < y$, and x, y are not the end points of one of the intervals removed from $[0, 1]$ to obtain C, then $f(x) < f(y)$.
      iii. If $x, y \in C$, $x < y$, and x, y are end points of one of the intervals removed from $[0, 1]$ to obtain C, then show that $f(x) = f(y) = \frac{p}{2^k}$ for some $p, k \in \mathbb{N}$ and p not divisible by 3. (Hint: If x is an end point of one of the intervals removed to obtain C, then $x = \frac{p}{3^k}$ for some $p, k \in \mathbb{N}$ and p not divisible by 3. Use (1) and 2(a) to obtain the result.)

iv. Extend f to a map from $[0,1]$ onto itself by defining its value on each interval missing from C to be its value at the end points. Show that f is continuous but not absolutely continuous (Hint: $f' = 0$ *a.e.*).