Due: September 8, 2008

Problems to be turned in : 2,5,6

1. Let $(\Omega, \mathcal{F}, \mu)$ be a measurable space and $\{f_n\}$ be a sequence of real-valued measurable functions.
   (a) Show that $\liminf_{n\to\infty} f)n$ is measurable.
   (b) Show that $\limsup_{n\to\infty} f)n$ is measurable.
   (c) Suppose f is such that $f_n \to f$ then show that f is measurable.

2. Prove or Disprove: Suppose $(\Omega, \mathcal{F}, \mu)$ be a measurable space, $f, \{f_n\}$ be a sequence of non-negative measurable functions and $f_n \to f$. Then $\int f_n d\mu \to \int f d\mu$.

3. Prove MCT in the case when f_n and f are non-negative extended real valued measurable functions. (Hint: Let $E_n = \{f_n = \infty\}, E = \{f = \infty\}$. Case 1: If $\mu(E_n) > 0$ for some n, then both sides of the desired identity are clearly infinite. Case 2: If $\mu(E_n) = 0\ \forall n$, $\mu(\Omega \setminus E) = 0$, then apply MCT to $f_n 1_E \wedge N$ and let $N \to \infty$. In all other cases show that the desired identity is a consequence of MCT proved in class.)

4. Let $(\Omega, \mathcal{F}, \mu)$ be a measurable space and f be a measurable function. If $f \geq 0$ a.e. then show that $\int f d\mu \geq 0$.

5. Let f be a non-negative measurable function defined on the measure space $(\Omega, \mathcal{B}, \mu)$. Define

$$\mu_f : \mathcal{B} \to [0, \infty] \text{ by } \mu_f(E) = \int_E f d\mu.$$

   show that: (i) μ_f is a measure defined on $\mathcal{B}$.
   (ii) μ_f is σ-finite if and only f is finite almost everywhere.
   (iii) $E \in \mathcal{B}, \mu(E) = 0 \Rightarrow \mu_f(E) = 0$.
   The measure μ_f is sometimes called the 'indefinite integral of f'.

6. Prove that if $f \in \mathcal{L}^1(\Omega, \mathcal{B}, \mu)$ the following are equivalent:
   (a) $\int_E f d\mu = 0$ for all $E \in \mathcal{B}$.
   (b) $f = 0\ \ \mu\ \ a.e.$