Due: September 1, 2008
Problems to be turned in : 1,2,3,7

1. If $T : X \to X'$ is a continuous map between topological spaces, then T is $(\mathcal{B}_X, \mathcal{B}_{X'})$ - measurable, where we write $\mathcal{B}_X$ to denote the σ-algebra generated by all open sets in X.

2. If $\mathcal{B}_i$ is a σ-algebra of subsets of a set X_i, and if $f_i : X \to X_i$ are functions, for each $i \in I$ - where I is some index set - show that there is a smallest σ-algebra $\mathcal{B}$ of subsets of X such that f_i is $(\mathcal{B}, \mathcal{B}_i)$-measurable for each $i \in I$. If we denote this σ-algebra $\mathcal{B}$ by $\sigma(\{f_i : i \in I\})$, show that

$$\sigma(\{f_i : i \in I\}) = \sigma(\{f_i^{(-1)}(U_i) : U_i \in \mathcal{S}_i \forall i \in I\}) ,$$

   for any family $\{\mathcal{S}_i : i \in I\}$ such that $\sigma(\mathcal{S}_i) = \mathcal{B}_i \ \forall i$.

3. If $X = \prod_{i \in I} X_i$, where $(X_i, \mathcal{U}_i)$ are topological spaces, if X is endowed with the product topology $\mathcal{U}$, verify that $\mathcal{B}_X = \sigma(\{\pi_i : i \in I\})$, where $\pi_i : X \to X_i$ is the natural projection map.

4. Let $f : \Omega \to \mathbb{R}^n$ be a function. Then there exist unique functions $f_i : \Omega \to \mathbb{R}, 1 \leq i \leq n$, such that $f(w) = (f_1(w), f_2(w), \ldots, f_n(w))$
   $\forall w \epsilon \Omega$.

5. If $\mathcal{S} = \{\prod_{i=1}^n (a_i, b_i) : -\infty < a_i < b_i < \infty \quad \forall i\}$ is the set of open rectangular boxes in $\mathbb{R}^n$, show that $\mathcal{B}_{\mathbb{R}^n} = \sigma(\mathcal{S})$.

6. If $(\Omega, \mathcal{B})$ is a measurable space, and if $f, f_1, \ldots, f_n$ are as in (1) above, show that f is $(\mathcal{B}, \mathcal{B}_{\mathbb{R}^n})$-measurable if and only if f_i is $(\mathcal{B}, \mathcal{B}_{\mathbb{R}})$-measurable for each $i = 1, 2, \ldots, n$.

7. If $f, f_2 : \Omega \to \mathbb{R}$ are measurable, then show that each of the following functions (from Ω to $\mathbb{R}$) is measurable :
   (a) $f_1 \vee f_2 = \max\{f_1, f_2\}$;
   (b) $f_1 \wedge f_2 = \min\{f_1, f_2\}$

8. Make precise the notion of measurability for an extended real-valued function - i.e. a function which takes values in $\bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$.