Due: August 25th, 2008

Problems to be turned in : 2,4,5,6

1. Let $\mathcal{S}$ denote the collection of intervals of the form (a,b], where $-\infty \leq a < b \leq \infty$ (where of course $(a, \infty]$ is to be interpreted as (a, ∞)). Let $\mathcal{A}$ denote the algebra $\mathcal{A}(\mathcal{S})$. Let $\mathbb{Q}$ denote the set of rational numbers in $\mathbb{R}$, and let $\mathcal{A}_0 = \mathcal{A} \cap \mathbb{Q}$. Consider the measure μ defined on $\mathcal{A}_0$ by $\mu(A) = 0$ or ∞ according as $A = \emptyset$ or $A \neq \emptyset$. Show that there exist more than one measure on $\sigma(\mathcal{A}_0)$ (in $\mathbb{Q}$) which agree with μ on $\mathcal{A}_0$.

2. Suppose (Ω , $\mathcal{B}$, μ) is a finite measure space, and suppose $\mathcal{A}$ is an algebra of subsets of Ω such that $\mathcal{B} = \sigma(\mathcal{A})$. Show that if $B \in \mathcal{B}$, and if $\varepsilon > 0$, then there exists $A \in \mathcal{A}$ such that $\mu(A \Delta B) < \varepsilon$, where $A\Delta B = (A - B) \cup (B - A)$. (Hint : Show that the collection of sets B in $\mathcal{B}$ for which the desired conclusion holds is a monotone class containing $\mathcal{A}$.)

3. Let (Ω, $\mathcal{B}$, μ) be a σ-finite measure space, let $\mathcal{A}$ be any algebra of subsets of Ω such that $\mathcal{B} = \sigma(\mathcal{A})$, and let $\mu_0 = \mu|\mathcal{A}$.

   (i) Show that $\mu_0^* = \mu^*$ (as functions on 2^{Ω}) and that in fact
   $$\mu_0^*(E) = \mu^*(E) = inf\{\mu(B) : B \in \mathcal{B}, E \subseteq B\}$$
   for all subsets E of Ω.

   (ii) If $N \subseteq \Omega$ and $\mu^*(N) = 0$, show that $N \in \mathcal{M}(\mu)$; we say that $(\Omega, \mathcal{M}(\mu), \mu^*)$ is a 'complete' measure space - meaning that if $N \subseteq M, M \in \mathcal{M}(\mu)$ and $\mu^*(M) = 0$, then $N \in \mathcal{M}(\mu)$; i.e., $\mathcal{M}(\mu)$ contains all μ^* - null sets.

   (iii) Show that $E \in \mathcal{M}(\mu)$ if and only if there exist $B_0, B_1 \in \mathcal{B}$ such that $B_0 \subseteq E \subseteq B_1$ and $\mu(B_1 - B_0) = 0$. (Hint : First assume $\mu(\Omega) < \infty$; the general case easily follows from this by the assumed σ-finiteness. Use (i) to lay hands on B_1. Define B_0 to be the complement of the B_1 you would have got for E'.)

   (iv) Make precise the statement that $(\Omega, \mathcal{M}(\mu), \mu^*)$ is '**the completion** of $(\Omega, \mathcal{B}, \mu)$.'

4. (a) Let $\mathcal{L} = \mathcal{M}(\mu)$ denote the σ-algebra of μ^*- measurable sets - (Members of $\mathcal{L}$ are called **Lebesgue measurable** sets.) Then, show that : $L \in \mathcal{L}, x \in \mathbb{R} \Rightarrow L + x \in \mathcal{L}$ and that $\mu^*(L + x) = \mu^*(L)$. (Hence μ^* is a translation invariant measure defined on $\mathcal{L}$.)

   (b) Let $I = (0, 1]$, and for $x, y \in I$ say $x \sim y$ if $x - y$ is a rational number. For $x, y \in (0, 1]$, define $x \overset{\circ}{+} y$ to be the unique element of (0,1] such that $(x + y) - (x \overset{\circ}{+} y)$ is an integer.

   (i) Show that $\sim$ is an equivalence relation on I, and that if $x, y \in (0, 1]$, then $x \sim y$ if and only if there exists a rational number $r \in (0, 1]$ such that $x = y \overset{\circ}{+} r$.

   (ii) If $[x]$ denotes the equivalence class of an element x of I - so $[x] = \{y \in I : x \sim y\}$ - let E be a set obtained by picking one element from each distinct equivalence class. Let $\{r_n\}_{n=1}^{\infty}$ be an enumeration of the rational numbers in $(0, 1]$, and define
   $$E_n = E \overset{\circ}{+} r_n = \{x \overset{\circ}{+} r_n : x \in E\}.$$
   Show that $(0, 1] = \coprod_{n=1}^{\infty} E_n$; also show that if $E \in \mathcal{L}$, then $E_n \in \mathcal{L}$ and $\mu^*(E_n) = \mu^*(E)$ for all n; deduce that $E \notin \mathcal{L}$.

5. Let $I = [0, 1]$. Let $I_1 = I_{11} = (\frac{1}{3}, \frac{2}{3})$ be the open middle third interval of I. Next, let $I_{21} = (\frac{1}{9}, \frac{2}{9})$ and $I_{22} = (\frac{7}{9}, \frac{8}{9})$ be the two open middle third intervals of $I - I_1$. Let $I_2 = I_{21} \cup I_{22}$. For $j \geq 3$ and $k = 1, 2, 3 \ldots, 2^{j-1}$, let I_{jk} be the open middle third intervals of $I - \cup_{k=1}^{j-1} I_k$ and let $I_j = \cup_{k=1}^{2^{j-1}} I_{jk}$. Finally, let $C = I - \cup_{j=1}^{\infty} I_j$. C is called the **Cantor set** .

(a) Show that C is compact and uncountable.

(b) Show that $\lambda(C) = 0$, where λ is lebesgue measure on $[0,1]$.

6. (a) If μ is a probability measure defined on the Borel σ- algebra $\mathcal{B}$ of $\mathbb{R}$, define $F : \mathbb{R} \to [0,1]$ by $F(x) = \mu((-\infty, x])$, and verify that

   (i) F is monotonically non-decreasing - i.e. $x \leq y \Rightarrow F(x) \leq F(y)$ - and right continuous - i.e., $\lim_{y \downarrow x} F(y) = F(x)$;

   (ii) F is discontinuous at x if and only if $\mu(\{x\}) > 0$; and

   (iii) $\lim_{x \to \infty} F(x) = 1, \lim_{x \to -\infty} F(x) = 0$.

   The function F is referred to as the **distribution function** of μ.

   (b) Conversely, if $F : \mathbb{R} \to [0,1]$ is a function satisfying (i) and (iii) above, (imitate the construction of Lebesgue measure to) show that there exists a unique probability measure μ on $\mathbb{R}$ such that $\mu((-\infty, x]) = F(x)$ for all x in $\mathbb{R}$.

   (c) Generalise (a) and (b) above to the case of σ-finite (rather than just probability) measures.