Problems to be turned in: 1(d), 3,4
Due: November 12th, 2008

1. Suppose μ is a σ-finite measure and λ, η, ν are arbitrary measures (real, complex or σ-finite) on the measurable space $(\Omega, \mathcal{B})$.

   We will say that ν is absolutely continuous with respect to μ if for all $E \in \mathcal{B}$, $\mu(E) = 0$ implies that $\mu(E) = 0$. This is denoted by $\nu << \mu$.

   (i) Show that $|\nu| << \mu$

   (ii) If $\lambda << \mu$ and $\nu << \mu$ then $\lambda + \nu << \mu$

   We will say that λ, ν are mutually singular if there is partition of $\Omega = \Omega_1 \coprod \Omega_2$ such that $\lambda(\Omega_1) = 0 = \nu(\Omega_2)$. This is denoted by $\lambda \perp \nu$

   (a) Show that $|\lambda| \perp |\nu|$

   (b) If $\eta \perp \nu$ and $\lambda \perp \nu$ then $\lambda + \eta \perp \nu$

   (c) If $\lambda << \mu$ and $\nu \perp \mu$ then $\lambda \perp \nu$

   (d) If $\lambda << \mu$ and $\lambda \perp \mu$ then $\lambda = 0$

2. Suppose $(\Omega, \mathcal{B}, \mu)$ is a finite measure space and $p \geq 1$. Suppose $f : \Omega \to \mathbb{C}$ is a bounded measurable function. Show that there is a sequence of simple functions $\{s_n\}_{n=1}^{\infty}$ such that $\| s_n - f \|_p \to 0$. Further if $g \in L^p(\Omega, \mathcal{B}, \mu)$ then for every $\epsilon > 0$ there exists an $f : \Omega \to \mathbb{C}$, bounded measurable function, such that $\| f - g \|_p < \epsilon$.

3. Suppose ν is a finite positive measure and λ is an arbitrary measure on a measurable space $(\Omega, \mathcal{B})$. Then show that there is atmost one pair of complex measures λ_1, λ_2 such that
$$\lambda_1 + \lambda_2 = \lambda, \lambda_1 << \mu, \lambda_2 \perp \mu$$
   (*Note: Existence was shown in class*)

4. Suppose ν is a σ-finite positive measure and λ is an arbitrary measure on a measurable space $(\Omega, \mathcal{B})$. Then show that there is a unique pair of complex measures λ_1, λ_2 such that
$$\lambda_1 + \lambda_2 = \lambda, \lambda_1 << \mu, \lambda_2 \perp \mu$$